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Low Reynolds number flow due to impulsively started circular cylinder

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SUMMARY

The authors obtain solutions for the title problem when the cylinder is imparted a constant velocity or a constant acceleration. It is assumed that initially both the solid obstacle and the infinite expanse of liquid surrounding it are at rest. The velocity and acceleration imparted to the cylinder are of finite magnitude, rectilinear and in a direction perpendicular to the cylinder axis. Both solutions are expressed by means of three distinct matched expansions. These analyses are valid as long as the Reynolds number, Re , is small.

Early in the processes under discussion and throughout the exterior of the cylinder, both flows are unsteady Stokesian to within a small error. Later they are represented by inner and outer expansions. The solution structure and the nature of the expansions suggest that close to the cylinder the effects of viscosity and transiency dominate the flow field. It is only in the outer fields, and some time after motion has commenced that the effect of vorticity-convection plays a significant role. For the case in which a steady rectilinear velocity is imparted to the cylinder, both the inner and outer expansions derived here approach those which were obtained previously by Proudman and Pearson for a steady flow pattern. From the leading terms in the two expansions representing the flows close to the cylinder, approximate expressions for the time-dependent drag are obtained.

1. Introduction

The authors present solutions for the unsteady fluid motion in the infinite expanse outside a solid circular cylinder. Initially, both the liquid and the solid are at rest. Suddenly a finite rectilinear velocity V, or a constant acceleration α is imparted to the cylinder in the direction perpendicular to its axis. The Reynolds number Re, which is based on that velocity or acceleration, the radius of the cylinder a and the kinematic viscosity ν is assumed to be small. Using this feature, the authors develop solutions of the matched asymptotic expansions type which represent the space and time dependence of the flows throughout the exterior of the cylinder. In principle, these hold for all times and the solution for case (a), where a constant velocity is imparted to the cylinder, shows that steady state is eventually attained. However, the applicability of the solution for case (b), where the obstacle is continuously accelerated, is limited because the small Re perturbations type of solution does not admit large obstacle velocity. The latter treatment is nevertheless of interest.

This work is the second in a sequence. The case of a sphere departing instantaneously from rest and acquiring a constant velocity has already been solved by the authors [1]. In terms of approach and technique the treatments of both geometries are very much alike and hence many of the arguments and algebraic details brought into [1] will not be repeated here. However, the solutions presented herewith differ from the three-dimensional one in their basic structure. This distinction is described, and to some extent explained, by considering the vorticity–balance relation.

In the cases treated in both works, vorticity is continuously generated at the obstacle's surface. Shortly after the processes have commenced the total amount that has been produced is small, and it has not yet diffused away into the far field, where the velocities of the liquid with respect to the sources of disturbance are high. Consequently, in both the two- and three-dimensional flows, initially the effect of convection is everywhere negligible. It follows, therefore, that for small time these flows can be mathematically expressed throughout by the solutions of the time-dependent Stokes equation.

As the processes go on, a significant amount of vorticity diffuses away from the obstacles. There, in the far field, the velocities with respect to the obstacles are relatively large. Thus, for large time and away from the obstacles, the unsteady Oseen equation, which was derived in [1], governs also the disturbance flows in the two-dimensional cases under discussion. That relationship expresses the balance between the temporal changes in the vorticity and the transport of this quantity by diffusion and convection, where the latter effect is represented by a linear term. In the treatment presented in [1] and in the analysis of case (*a*), the solutions of that equation asymptotically approach the steady outer solutions previously obtained by Proudman and Pearson [2] for a sphere and a cylinder, respectively.

But this is also where the similarity between these two works ceases. One of the important results derived in [1] is that, to the highest order, the flow close to the sphere is always independent of the effect of convection. This does not hold if higher-order terms are included [3], but if one restricts oneself to the unsteady approximation, then the unsteady Stokesian solution, which initially holds (approximately) throughout the exterior of the sphere, holds also close to it not only initially but forever. However, the flows close to the translating cylinder are (approximately) independent of the effect of convection only when motion commences. Late in the processes, the unsteady Stokesian solutions hold no longer and the stream functions are given by Proudman and Pearson's [2] inner-expansion form:

$$\psi \sim \Delta\,[(r\ln r - r/2 + (2r)^{-1})\sin\theta] + O(\Delta^2),$$

$$\psi \sim \Delta Re(3t/2)\,[(r\ln r - r/2 + (2r)^{-1})\sin\theta] + O(\Delta^2)$$

for cases (*a*) and (*b*), respectively. Here (r, θ) are the normalized polar coordinates and t designates time. These expressions reflect the indirect effect of convection, although this may not be immediately apparent. Indeed, the expressions between the square brackets satisfy the biharmonic equation. This implies that at every point in the inner field, the effect of vorticity-convection is negligible compared with that of diffusion. However, the amplitude of the inner field is determined by matching with the outer region, where convection has been shown to be significant. This influence is reflected by Proudman and Pearson's gauge function Δ, which depends on Re. In other words, the effect of vorticity-convection, which is prevalent for large times and away from the obstacles, does not significantly affect the neighbourhood of the sphere considered in [1] but does affect the flow pattern in the vicinity of the cylinder.

As presented, the description of the three different physical behaviours in the various sub-domains should be taken as mere conjectures. However, the coordinate scaling, which is carried out in the next two sections, is based on these conjectures. This particular scaling affects, in turn, the forms of the governing equation and hence also the different expansions representing the two solutions under discussion. Consequently, the fact that the expansions match provide reassurance that the picture described and the conjectures made are plausible and that the three-expansion solutions are indeed valid.

2. Case (*a*), constant velocity

Denoting dimensional variables by primes, the normalized stream function, and time and space coordinates are defined as follows

$$\psi'/Va \equiv \psi, \quad t'\nu/a^2 \equiv t, \quad (x',y')/a \equiv (x,y). \tag{1}$$

Thus, the governing equation reads

$$\frac{\partial}{\partial t}(\nabla^2\psi) - Re\,\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)} = \nabla^4\psi \tag{2}$$

in which ∇^2 is the Laplacian. The origin of the cartesian (x, y) coordinate system is fixed to the centre of the cylinder. Therefore, the boundary and initial conditions are given by

$$\psi = \partial\psi/\partial r = 0, \quad t \geq 0, \quad r = 1, \quad 0 \leq \theta \leq 2\pi; \qquad (3), (4)$$

$$\psi \sim r\sin\theta, \quad t > 0, \quad r \to \infty, \quad 0 \leq \theta \leq 2\pi; \tag{5}$$

$$\psi = 0, \quad t = 0, \quad 1 < r < \infty, \quad 0 \leq \theta \leq 2\pi. \tag{6}$$

The polar and cartesian coordinate systems used here are interrelated in the usual manner and the centres of both systems coincide. It follows that the cylinder moves in the negative x-direction with respect to the fluid at infinity, which is at rest.

As explained in the Introduction, the flow under discussion exhibits three distinct physical behaviours: early in the process throughout the field, late but away from the obstacle, and late in the vicinity of the cylinder. The solution for ψ representing the flow is accordingly expressed by three different expansions. These are designated by $\psi^{(e,t)}$, $\psi^{(l,o)}$ and $\psi^{(l,i)}$. The first of each superscript-pair indicates whether the expansion holds 'early' or 'late'. The second suggests 'inner', 'outer' or 'throughout'. Note that it is assumed that the regions of validity of every two expansions overlap, so that the matching principle can be invoked.

The space and time variables associated with each sub-domain reflect the prevalent physical behaviour. As explained, early in the process the effect of convection is nowhere significant. Vorticity is transported mainly by diffusion and hence V does not; yet ν and a do play a role in the scaling of the variables characterizing the (e, t) sub-domain. Consequently, (r, t), as defined, serve as the space and time coordinates and relationship (2) is the governing equation there.

Another rather obvious feature of the scaling scheme is that it should make the expression for the flow far from the cylinder independent of the radius a. Note that although in principle $\psi^{(e,t)}$ depends on r and t and, although a plays a role in the scaling of these, far from the obstacle the suitably dimensionalized solution is indeed independent of a. The leading term in this expansion, which is derived below, consists of two components. The component y, which represents a uniform stream, becomes independent of a when multiplied by the normalizing factor Va. The other component evidently decays as r increases. However, the term in the expansion $\psi^{(l,o)}$, which represents the disturbance flow and the wake behind the cylinder, leaves a trace at infinity. It follows that in the scaling of the independent variables of the (l, o) sub-domain, a can play no role. By elimination, only ν and V do, and hence the space and time coordinates should be scaled as follows:

$$R = r\,Re, \quad T = t\,(Re)^2. \tag{7}$$

Thus, the length scale is (V/ν) and this is known as the viscous length. The time scale (ν/V^2) was encountered in [1]. Finally, close to the obstacle and late in the process, the characteristic length is again a. However, the time variations are introduced via matching with the solution prevalent in the late-outer sub-domain. Hence, the coordinates characterizing variations in the (l, i) sub-domain are (r, T).

The expansion which holds early in the process and throughout may be written as

$$\psi = \psi^{(e,t)} \sim \psi_0{}^{(e,t)} + (Re)\,\psi_1{}^{(e,t)}. \tag{8}$$

It follows from equation (2) that the leading term $\psi_0{}^{(e,t)}$ must satisfy the following relationship

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\,\psi_0{}^{(e,t)} = 0. \tag{9}$$

Using both the initial and the boundary conditions (3)–(6), one finds that the Laplace transform of $\psi_0{}^{(e,t)}(r, \theta, t)$, defined thus:

$$\Psi_0{}^{(e,t)}(r, \theta, s) \equiv \int_0^\infty \exp(-st)\,\psi_0{}^{(e,t)}(r, \theta, t)\,\mathrm{d}t, \tag{10}$$

is given by

$$\Psi_0{}^{(e,t)}(r, \theta, s) = \left[\frac{2}{s^{3/2}}\,\frac{1}{K_0(\sqrt{s})}\left\{K_1(\sqrt{s}\,r) - \frac{1}{r}\,K_1(\sqrt{s})\right\} + \frac{1}{s}\left(r - \frac{1}{r}\right)\right]\sin\theta, \tag{11}$$

where K_0 and K_1 are the modified Bessel functions of the second kind.

In order to find the expansion form $\psi^{(l,i)}$ one has to adopt the standard technique of recasting the inverse of the solution (11) in terms of the coordinates associated with the (l, i) sub-domain, namely (r, T). It was shown in [1] that in view of the relationship $T = (Re)^2 t$, the T-transform of the expansion sought can also be obtained by letting Re vanish in the following relationships

$$\Psi^{(l,i)}(r,S) \equiv \int_0^\infty \exp(-ST)\,\psi^{(l,i)}(r,T)\,\mathrm{d}T = Re^2\,\Psi^{(e,t)}(r,SRe^2), \tag{12}$$

$$SRe^2 = s. \tag{13}$$

Therefore, $K_0(\sqrt{s})$ is expressed as follows

$$K_0(\sqrt{s}) = K_0(\sqrt{S}\,Re) \sim -(\gamma + \ln\sqrt{S} - (\ln Re^{-1} + k) + k - \ln 2)) + O(Re), \tag{14}$$

where γ is the Euler constant, and after rather lengthy manipulations, one gets the following result

$$\Psi^{(l,i)} \sim \frac{\Delta}{Re^2 S}\,(r\ln r - r/2 + 1/2r)\sin\theta + O(\Delta^2). \tag{15}$$

Here Δ is the gauge function obtained by Proudman and Pearson [2] which is defined thus

$$\Delta = (\ln(1/Re) + k)^{-1}. \tag{16}$$

At this stage of the analysis, k need not be determined, Kaplun and Lagerstrom [4] suggested that if one is to consider higher-order approximations, then there are advantages to the choice $k = \ln 4 + 1/2 - \gamma$.

Evidently, the result (15) expresses not only the form of the expansion, but also the solution for the leading term of the T-transform $\Psi^{(l,i)}$. Indeed, by rewriting the governing equation in terms of (r, T) one gets

$$Re^2\,\frac{\partial(\nabla^2\psi)}{\partial T} + Re\,\frac{\partial(\psi,\nabla^2\psi)}{(x,y)} = \nabla^4\psi. \tag{2'}$$

This implies that the leading term in the expansion $\psi^{(l,i)}$ should be in the form of a product of a function of T times a solution of the biharmonic equation which satisfies the conditions (3) and (4). The result (15) meets these requirements. Moreover, since $(SRe^2)^{-1}$ or s^{-1} is the transform of unity, it follows that for $T \to \infty$ the inverse solution reduces to Proudman and Pearson's [2] expression for the steady near field. Its matching with the expression $\psi^{(l,o)}$ for all other values of T will be also verified below.

Recasting now the governing equation in terms of the variables associated with the (l, o) sub-domain, one gets

$$\frac{\partial}{\partial T}(\overline{\nabla}^2\psi) - Re\,\frac{\partial(\psi,\overline{\nabla}^2\psi)}{\partial(X,Y)} = \overline{\nabla}^4\psi, \tag{2''}$$

where $(X, Y) = R(\cos\theta, \sin\theta)$ are the late-outer-field cartesian coordinates and $\overline{\nabla}^2$ is the Laplacian in terms of these. It is assumed that the solution in that sub-domain has the form

$$\psi^{(l,o)} \sim Y/Re + (\Delta/Re)\,\psi_1^{(l,o)}(R,\theta,T), \tag{17}$$

so that the term representing the disturbance in the uniform flow is governed by the unsteady Oseen equation which is very similar to that derived in [1]:

$$\left[\frac{\partial}{\partial T} + \frac{\partial}{\partial X} - \overline{\nabla}^2\right] \overline{\nabla}^2 \psi_1{}^{(l,o)} = 0. \tag{18}$$

Clearly $\psi_1{}^{(l,o)}$ is the solution of an initial-value problem. It must therefore satisfy conditions at each end for all values of T as well as an initial condition for all values of R. Since the solution for $\psi^{(e,t)}$ for large r is evidently irrotational, $\overline{\nabla}^2 \psi^{(l,o)}$ is taken to be zero initially and therefore by Laplace transforming equation (18) T-wise, one gets

$$(S + \partial/\partial X - \overline{\nabla}^2)\,\overline{\nabla}^2 \Psi_1{}^{(l,o)} = 0. \tag{19}$$

The integration of equation (19) is achieved as follows. Recasting in terms of $R = rRe$ the T-wise Laplace transform of the vorticity associated with the solution (15) one gets

$$\nabla^2 \Psi_0{}^{(l,i)} = (Re)^2 \overline{\nabla}^2 \Psi_0{}^{(l,i)} \sim (Re)\,\Delta\,\frac{2}{SR}\sin\theta. \tag{20}$$

Since $\nabla^2 \psi_1{}^{(l,o)}$ must match this expression, a solution of equation (19) is sought which is proportional to $\sin\theta$. The most general expression which satisfies this requirement and is finite at infinity, is

$$\overline{\nabla}^2 \Psi_1{}^{(l,o)} = P(S)\exp(X/2)\,K_1(\zeta R/2)\sin\theta, \tag{21}$$

where

$$\zeta = (4S + 1)^{1/2}. \tag{22}$$

The integration constant $P(S)$ is evaluated by invoking the requirement that the two expressions for the vorticity should match. This yields

$$P(S) = \zeta/S. \tag{23}$$

Note that this matching holds for every value of S and hence for the entire time span $0 < T < \infty$.

The solution for the transform of the stream function representing the disturbance flow in the (l, o) sub-domain is evidently given by

$$\Psi_1{}^{(l,o)}(X, Y, S) = -\frac{2}{S}\frac{\partial}{\partial Y}\int_0^\infty \exp(-S\xi)\{\exp(\widetilde{X}/2)\,K_0(\zeta\widetilde{R}/2) + \ln\widetilde{R}\}\,d\xi, \tag{24}$$

where $\widetilde{X} = X - \xi$, $\widetilde{R}^2 = \widetilde{X}^2 + Y^2$. This result is derived in Appendix A. Note that the solution (24) in effect consists of two line integrals. One represents the summation of irrotational doublets distributed along the positive X-axis. The other represents a similar distribution but of the solution of the equation

$$(S + \partial/\partial X - \overline{\nabla}^2)\, G(X, Y, S) = 0 \tag{25}$$

which vanishes at infinity and has the following singular behaviour at the origin:

$$G \sim -\, Y(X^2 + Y^2)^{-1} \quad \text{as } (X, Y) \sim 0. \tag{26}$$

The singularities due to the two distributions on the X-axis cancel out and the solution (24) is therefore continuous across this axis.

3. Case (b), constant acceleration

As explained, the solution of this case is also assumed to posses distinct behaviours in the three sub-domains (e, t) (l, o) and (l, i). It is, again, anticipated that early in the process throughout the exterior of the cylinder, a and a^2/ν are the characteristic length and time scales. However, the characteristic velocity in this case is $(\alpha a)^{1/2}$ rather than V and the stream function must, therefore be normalized with respect to $(\alpha a^3)^{1/2}$. It is thus found that the boundary conditions (3) and (4), the initial condition (6) and the governing equation (2) hold here too except that the Reynolds number is defined thus:

$$Re \equiv (\alpha a^3)^{1/2}/\nu. \tag{27}$$

The condition imposed on the flow at infinity reads

$$\psi \to (Re)\, tr \sin\theta, \quad t > 0, \quad r \to \infty, \quad 0 \le \theta \le 2\pi. \tag{5$'$}$$

Therefore, the leading term in the expansion which holds early throughout

$$\psi = \psi^{(e,t)} \sim (Re)\, \psi_1{}^{(e,t)} + (Re)^2\, \psi_2{}^{(e,t)} \tag{28}$$

is of $O(Re)$. Its Laplace transform can be easily shown to be given by

$$\Psi_1{}^{(e,t)} = \frac{1}{s^2}\left[\left(r - \frac{1}{r}\right) + \frac{2}{\sqrt{s}\, K_0(\sqrt{s})}\left(\frac{K_1(\sqrt{s})}{r} - K_1(r\sqrt{s})\right)\right] \sin\theta, \tag{29}$$

which is obviously $(1/s)$ times the right hand side of equation (11).

As explained in the last section late in the process and far from the cylinder the length and time scales must be independent of a. But, since in the case under discussion the obstacle is imparted acceleration α rather than velocity V, the new viscous length scale is $(\nu^2/\alpha)^{1/3}$ and the corresponding time scale is $(\nu/\alpha^2)^{1/3}$. It follows that the independent variables typifying the late-outer domain (R, T) must be given by

$$R = r'(\alpha/\nu^2)^{1/3} = rRe^{2/3}, \quad T = t'(\alpha^2/\nu)^{1/3} = tRe^{4/3} \tag{30}$$

while, as before, the independent variables of the late-inner domain are (r, T).

Rather than derive the appropriate late-inner expansion, it will be verified that it is given by

$$\psi = \psi^{(l,i)} \sim \frac{3}{2} \frac{\Delta}{Re^{1/3}} T(r \ln r - r/2 + 1/2r) \sin\theta. \tag{31}$$

Recasting equation (2) in terms of (r, T), as defined by relationship (30), one finds that the term which represents the local temporal changes, becomes of $O(Re^{4/3})$ while vorticity convection is of $O(Re)$. It thus follows that the leading component in the late-inner expansion has to be in a form of a product of a biharmonic function with an arbitrary function of time. Such a form is indeed exhibited by the right hand side of equation (31). Moreover, the particular function of time is such that by setting $r = R\,Re^{-2/3}$ and letting Re approach zero, one finds that the late-outer expansion is given by

$$\psi^{(l,o)} \sim \frac{TR}{Re} \sin\theta + \frac{\Delta}{Re} \psi_1{}^{(l,o)}(R, T). \tag{32}$$

The first term indeed represents the accelerating uniform stream and that is the flow field far from the obstacle. Again, by repeating the manipulations involved in the derivation of the result (15), one can show that the s and S Laplace transforms of the leading term in $\psi^{(e,t)}$ and $\psi^{(l,i)}$ match. Note that in the present analysis s is replaced by $SRe^{4/3}$ and equation (12) is accordingly modified.

It is of physical interest to evaluate the leading terms in the expansions $\psi^{(e,t)}$ and $\psi^{(l,i)}$ – as the authors do – because, when combined, they yield an expression for the drag. However, it is as important to verify the conjectures made about the three-expansion structure of the solution and the corresponding behaviour of the flows under discussion. So far, it has been shown that the expansions thus constructed match when only the leading irrotational component of $\psi^{(l,o)}$ was accounted for, and this is clearly inadequate. Therefore an expression for the rotational term $\psi_1{}^{(l,o)}$, which represents the disturbance created by the cylinder, will now be obtained. It will be shown that, when included in $\psi^{(l,o)}$, this expansion matches the others. However, within the framework of this paper $\psi_1{}^{(l,o)}(R, \theta, T)$ will not be evaluated and plotted.

Recasting the governing equation in terms of the late-outer variables (R, T) one finds that equation (2″) holds in this case too. Then, by substituting expansion (32) into equation (2″), one finds that the disturbance stream function is governed by the following unsteady Oseen equation:

$$\left(\frac{\partial}{\partial T} + T \frac{\partial}{\partial X}\right) \nabla^2 \psi_1{}^{(l,o)} = \nabla^4 \psi_1{}^{(l,o)}. \tag{33}$$

This can be explained on physical grounds which are also applicable in [1] and in the treatment of case (a). But here that relationship is of a somewhat different form. The term representing vorticity-convection reflects the linear growth of the velocity of the stream at infinity with time. This complicates matters considerably. In particular it is impossible to solve equation (33) by manipulating the result (24) although, noting that the results (11) and (29) as well as (15) and (31) are algebraically similar, one would be tempted to try to look for a shortcut.

As in case (a), $\overline{\nabla}^2 \psi_1{}^{(l,o)}$ is taken to be initially zero. Hence, by Laplace transforming equation (33) one gets

$$\left(S - \frac{\partial^2}{\partial S \partial X} - \overline{\nabla}^2\right) \overline{\nabla}^2 \Psi_1{}^{(l,o)} = 0. \tag{34}$$

It can be verified that this admits solutions of the form

$$\overline{\nabla}^2 \Psi_1{}^{(l,o)} = \begin{Bmatrix} Ai \\ Bi \end{Bmatrix} \left(\frac{X + \beta + 2m^2}{2^{1/3}}\right) \begin{Bmatrix} Ai \\ Bi \end{Bmatrix} \left(\frac{X - 2S + \beta}{2^{1/3}}\right) \sin(mY). \tag{35}$$

Here Ai and Bi are Airy integrals, as defined in Jeffreys and Jeffreys' [5] well-known text. In so far as the governing equation is concerned, all four possible products are admissible when β and m can attain any value. But in view of the other requirements imposed on the vorticity, the solution has to be of the form:

$$\overline{\nabla}^2 \Psi_1{}^{(l,o)} = \int_{-\infty}^{\infty} \int_0^{\infty} F(\beta)(2\pi)2^{1/3} m$$
$$\begin{Bmatrix} Bi\left(\frac{X + 2m^2 + \beta}{2^{1/3}}\right) Ai\left(\frac{2m^2 + \beta}{2^{1/3}}\right) Ai\left(\frac{X - 2S + \beta}{2^{1/3}}\right) \\ Bi\left(\frac{2m^2 + \beta}{2^{1/3}}\right) Ai\left(\frac{X + 2m^2 + \beta}{2^{1/3}}\right) Ai\left(\frac{X - 2S + \beta}{2^{1/3}}\right) \end{Bmatrix} \sin(mY)\, \mathrm{d}m\, \mathrm{d}\beta \tag{36}$$

for $x \lessgtr 0$. $F(\beta)$ is determined below.

It is first noted the two expressions are equal for $X = 0$. Hence, the Laplace transform of the vorticity is continuous along the Y-axis. The X derivatives obtained from these two expressions are also equal there except at the origin. To show that this is so one writes the difference between the two thus,

$$2^{1/3}(2\pi) \int_{-\infty}^{\infty} \int_0^{\infty} F(\beta)\, m\, Ai\left(\frac{\beta - 2S}{2^{1/3}}\right) \left[Ai'\left(\frac{2m^2 + \beta}{2^{1/3}}\right) Bi\left(\frac{2m^2 + \beta}{2^{1/3}}\right) - Bi'\left(\frac{2m^2 + \beta}{2^{1/3}}\right) Ai\left(\frac{2m^2 + \beta}{2^{1/3}}\right)\right] \sin(mY)\, \mathrm{d}m\, \mathrm{d}\beta.$$

Evidently the expression in the square bracket is the Wronskian, and it is equal to π^{-1}. Therefore, in the resulting expression for the difference, a term in the form of a Fourier sine integral can be factored out. The latter and hence also the entire expression vanishes everywhere except at $Y = 0$. Both here and elsewhere in this work Fourier integrals are evaluated using Lighthill's [6] generalized-functions approach. Note also that since the Laplace transform of the vorticity satisfies a second-order equation, the continuity of its zeroth and first derivatives everywhere except at the origin, implies that all higher derivatives are similarly continuous.

Since the outer solution for the vorticity (26) is in the form of a Fourier integral, to obtain its inner limit it is not enough to recast it in terms of the inner variables (x, y). As shown by Bentwich [7], one must also replace the integration parameter thus:

$$M = m\,Re^{-2/3}. \tag{37}$$

Broadly speaking, this means that the behaviour of the integral close to the origin in the physical plane is affected by the contribution due to integration over the outskirts in the Fourier-transform domain. That contribution can be approximated by replacing the Airy integrals by their asymptotic values for large positive arguments. Thus the inner limit of the outer solution is

$$\begin{aligned}\overline{\nabla}^2\Psi_1{}^{(l,o)} &\sim \frac{1}{Re^{2/3}}\int_{-\infty}^{\infty}\int_0^{\infty} F(\beta)Ai\left(\frac{\beta-2S}{2^{1/3}}\right)\sin(My)\cdot\\ &\cdot\exp\left\{\pm\frac{4}{3}\frac{M^3}{Re^2}\left[\left(1+\frac{\beta Re^{4/3}+xRe^2}{2M^2}\right)^{3/2}-\left(1+\frac{\beta Re^{4/3}}{2M^2}\right)^{3/2}\right]\right\}dM\,d\beta,\end{aligned} \tag{38}$$

where the plus and minus signs correspond to the first and second expressions, respectively, of equation (36). But with both signs, for vanishingly small Re, one gets

$$\overline{\nabla}^2\Psi_1{}^{(l,o)} \sim Re^{-2/3}\int_{-\infty}^{\infty} F(\beta)\,Ai\left(\frac{\beta-2S}{2^{1/3}}\right)d\beta\int_0^{\infty}\exp(-Mx)\sin(My)\,dM. \tag{39}$$

The integral with respect to M is clearly $y\,(x^2+y^2)^{-1}$, and it is shown in Appendix C that if $F(\beta)$ is suitably chosen, the integral with respect to β is $3S^{-2}$. Thus, the inner and outer expression for the Laplace transform of the vorticity can be made to match.

The Laplace transform of the disturbance stream function in the outer field late in the process is

$$\begin{aligned}\Psi_1{}^{(l,o)} = -(\pi)2^{1/3}\int_{-\infty}^{\infty}\int_0^{\infty} F(\beta)\{&\int_{-\infty}^{X}\Lambda(\xi,S,m,\beta)\ \exp[m(\xi-X)]\,d\xi\ +\\ &+\int_X^{\infty}\Lambda(\xi,S,m,\beta)\exp[m(X-\xi)]\,d\xi\}\sin(mY)\,dm\,d\beta.\end{aligned} \tag{40}$$

Here $\Lambda(X, S, m, \beta)$ is the top or bottom triple product on the right-hand side of equation (36) for negative or positive X, respectively. In fact the solution for $\Psi_1{}^{(l,o)}$ is obtained by integrating the latter relationship, and imposing the condition that the disturbance stream function should vanish at infinity. Since equation (36) is of the Poisson type, it follows that its solution $\Psi_1{}^{(l,o)}$ is unique. Therefore, if the vorticity of the inner field matches that associated with the outer, the corresponding stream functions match too. This can also be verified by recasting $\Psi_1{}^{(l,o)}$ in terms of x, y, M and η, where the latter integration variable is defined thus:

$$\xi = \eta Re^{2/3}. \tag{41}$$

Again approximating the Airy integrals by their asymptotic values and utilizing the results proved in Appendix C one gets for $x > 0$:

$$\Psi_1^{(i,o)} \sim Re^{4/3} \int_{-\infty}^{\infty} \int_0^{\infty} (1/2) F(\beta) Ai\left(\frac{\beta - 2S}{2^{1/3}}\right) \frac{1}{M} \sin(My) \cdot$$

$$\cdot \left\{ \exp(-Mx) \left[\int_{-\infty}^{0} \exp(2M\eta)\, d\eta + \int_0^x d\eta \right] + \right. \tag{42}$$

$$\left. + \exp(Mx) \int_x^{\infty} \exp(-2M\eta)\, d\eta \right\} d\eta =$$

$$= -\frac{3}{S^2} \frac{Re^{4/3}}{2} \int_0^{\infty} \left(\frac{1}{M^2} + \frac{x}{M}\right) \exp(-Mx) \sin(My)\, dM =$$

$$= Re^{4/3} \frac{3}{2} \frac{1}{S^2} (\ln r) y.$$

The same result is of course obtained for $X < 0$. Therefore by incorporating the last result with expansion (32) one can show that it matches expansion (31).

4. The drag and other results

As pointed out in the Introduction, the three-expansion structure of the solutions obtained reflects the relative effectiveness of vorticity-diffusion, vorticity-convection and temporal changes of that quantity in various parts of the space-time domain. Clearly, two different flow fields, namely those analyzed here, can be represented by one and the same form of solution, because they are physically similar in that sense. These are probably not the only two. It appears reasonable to expect that the solution structure proposed is applicable to other flows in which a cylinder immersed in liquid departs from rest. The manner of departure should be such that the relative effectiveness of convection, diffusion and temporal changes are as conjectured in the Introduction and verified by the fact that the expansions match. A necessary condition for such physical situation to occur is that the velocity of the cylinder is always finite and Reynolds number based on the maximum velocity is small.

While each of the three expansions obtained in the two analyses holds in a different subdomain, together each threesome covers the entire space and time domain under discussion. The same holds for the matched-asymptotic-expansions solution for a translating sphere which was presented in [1]. Hence, both works provide comprehensive physical pictures. However, if one is interested in the time-dependent drag, the construction of the complete solutions is of more crucial importance in the treatments of the two-dimensional cases at hand. Indeed, Villat [8] successfully calculated it for a sphere by ignoring the outer field and assuming that close to the translating obstacle the flow is unsteady Stokesian. Decades later, his assumption and the result based thereon were verified by the author's complete solution. However, it is impossible to apply Villat's short-cut to the cases at hand. The near field, from which the stresses on the cylinder are deduced, is expressed by two different expansions, $\psi^{(e,t)}$ and $\psi^{(l,i)}$. Moreover, the latter matches $\psi^{(l,o)}$ and thus affects it. Consequently, in order to calculate the drag, one must solve for all three expansions which cover the entire space and time domain.

The leading terms in the two expansions $\psi^{(e,t)}$, which were obtained in the solutions for cases (*a*) and (*b*), satisfy the same linear homogeneous equation. The same holds for the leading terms in $\psi^{(l,i)}$. It is for that reason that corresponding terms in the (e, t) and (l, i) expansions are simply related. Thus the Laplace transforms of $\psi_1{}^{(e,t)}$ and $\psi_1{}^{(l,i)}$ for case (*b*) are $1/s$ and $1/S$ times the corresponding transforms for case (*a*). Moreover, since the drag force depends on the flow field close to the obstacle, this similarity is reflected by the following rather simple and useful relationship

$$d_a(t) \simeq \int_0^t d_v(\hat{t})\, \mathrm{d}\hat{t}, \tag{43}$$

which follows from the properties of Laplace transforms. Here d_a and d_v are the drag forces experienced by the cylinder when it is instantaneously imparted a constant acceleration and a constant velocity, respectively. For different modes of departure from rest, expressions for the time-dependent drag can be obtained using the Laplace-transform properties as shown in [1]. Note that these results for various modes of departure are easily obtained only under drastic truncation. In principle, the solution for the flows differ. This difference is reflected, among other things, in the dissimilarity of the expressions for the late-outer disturbance flows for cases (*a*) and (*b*).

The process of integrating the stresses on the cylinder's surface is very similar to that carried out in the case of the sphere considered in [1]. Therefore, in order to save space, it will be stated here without proof that the Laplace transform of the drag forces on cylinders of unit axial length are

$$D_v(s) = \rho V^2 a \frac{\pi}{Re} \left(\frac{4}{s^{1/2}} \frac{K_1(\sqrt{s})}{K_0(\sqrt{s})} + 1 \right); \quad D_a(s) = \frac{1}{s} D_v(s). \tag{44}$$

These are based on the one-term approximate solutions for $\psi^{(e,t)}$, which hold for small time. These account for the added-mass effect which is prominent initially when the cylinder instantaneously departs from rest. The inverse of the s-independent contribution in the first relationship and the s^{-1} term in the second are, respectively, the delta function $\delta(t)$ and the Heaviside step function $H(t)$. There is a similar s-independent contribution to the right-hand side of equation (29) of [1], which expresses the added-mass contribution to the drag on a sphere.

The other important effects reflected in these solutions for $\psi^{(e,t)}$ and in the expressions for the drag (44) derived therefrom, are the diffusion and temporal changes of vorticity. But since the processes of vorticity and heat diffusion are governed by the same equation, it is possible to make use here of the solution for the transient temperature distribution throughout the exterior of the cylinder which is given in Carslaw and Jaeger's [9] text. Indeed, the Laplace transforms appearing there are identical, or easily reducible, to the terms involving K_0 and K_1 in equations (11) and (44). It can thus be shown that for case (*a*) the drag, as derived from the solution (11), is given by

$$d_v = \frac{\rho V^2 a\pi}{Re}\left\{\delta(t) + \frac{16}{\pi^2}\int_0^\infty e^{-\xi^2 t}\,\frac{d\xi}{\xi[J_0^2(\xi) + Y_0^2(\xi)]}\right\}. \tag{45}$$

For small values of t it is useful to use the approximation given in the text cited, namely

$$d_v = \frac{\rho V^2 a\pi}{Re}\,\{\delta(t) + (\pi t)^{-1/2} + 1/2\} + O(t^{1/2}), \qquad t \ll 1, \tag{46}$$

and obviously d_a can be obtained by straightforward integration.

It was shown that by setting $SRe^2 = s$ and $SRe^{4/3} = s$ in relationship (11) and (29) and letting Re approach zero, one does not only get the expansion forms of $\psi^{(l,i)}$, but also the correct expression for the highest-order term. It follows that Proudman and Pearson's [2] well-known expression for the steady-state drag is embedded in the form (45). Indeed, one gets it by setting $t = T/Re^2$, letting Re become small and retaining the highest-order term. This operation is carried out using Carslaw and Jaeger's [9] approximate relationship for the inverse of the Laplace transform which involves Bessel functions,

$$L^{-1}\,[(s^{-1/2}K_1(\sqrt{s})/K_0(\sqrt{s})] \sim \frac{2}{\ln 4 + \ln t - 2\gamma} - \frac{2}{(\ln 4 + \ln t - 2\gamma)^2}\,. \tag{47}$$

It clearly holds for the large argument, i.e. for a fixed value of T and small Re. Consequently, late in the process, d_v is indeed given by

$$d_v \sim (\rho V^2 a)\,(4\pi)\,\frac{\Delta}{Re} + O(\Delta^2/Re). \tag{48}$$

It is noted at this point that when the problem of steady flow past a cylinder was tackled, considerable ingenuity was required to discover the gauge function Δ with its logarithmic dependence on Re. For the case of a steady stream past a sphere the regular expansion form holds. Hence that solution provides no clues. However, by considering physically more complicated two-dimensional flow problems, one can actually deduce the functional relationship $\Delta(Re)$. This has been shown here, where unsteadiness was introduced. It was also shown by Bentwich [10] who considered the complications when the flow past the cylinder is bounded on one side.

The drag coefficient is defined in the conventional manner:

$$C_D \equiv d_v/\rho V^2 a. \tag{49}$$

It is plotted in Figure 1 as a function of time for various values of Re. These graphs embody both the results (45) and (48). Logarithmic scales are used in both the abscissa and ordinate so as to cover as wide a range as possible. But since with such scaling the origin cannot be included, this figure does not reflect the added-mass effect. The curves do exhibit the $t^{-1/2}$ singularity given by equation (47). Note also that as Re is increased C_D approaches its steady asymptotic value faster. Using relationship (43) one can derive the corresponding curves for case (*b*). Evidently, at the departure from rest the drag grows like $t^{1/2}$, while it increases linearly with time long after motion has commenced. The inversion of $\psi^{(e,t)}$ for case (*a*) can be easily arrived at by performing some algebraic manipulations on the recorded solution of Carslaw and Jaeger [9]. It may be written as

$$\psi^{(e,t)}(r,\theta,t)=$$

$$\left\{(r-1/r)+\frac{4}{\pi}\int_0^\infty(\mathrm{e}^{-\xi^2 t}-1)\,\frac{J_1(r\xi)Y_0(\xi)-Y_1(r\xi)J_0(\xi)-2/(\pi r\xi)}{J_0^2(\xi)+Y_0^2(\xi)}\,\frac{\mathrm{d}\xi}{\xi^2}\right\}\sin\theta. \tag{50}$$

Clearly $\psi^{(e,t)}$ vanishes for $r=1$ and has the following useful approximation for small time

$$\psi^{(e,t)}(r,\theta,t)\cong$$

$$(r-1/r)+\frac{2}{\sqrt{r}}\left[2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{(r-1)^2}{4t}\right)-(r-1)\operatorname{erfc}\left(\frac{r-1}{2\sqrt{t}}\right)-\frac{4}{r}\sqrt{\frac{t}{\pi}}\right]\sin\theta. \tag{51}$$

If presented graphically, it demonstrates the 'birth' of a pair of vortices on the cylinder when motion commences and the propagation of this pair outward and downstream as time progresses.

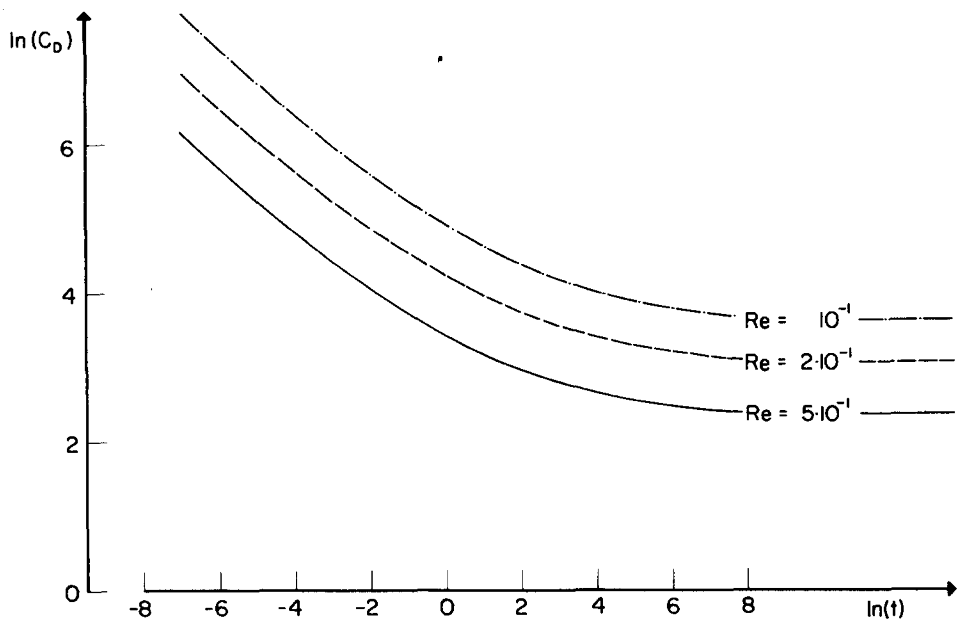

Figure 1. **Time-dependent drag coefficient.**

The graphical presentation is not given herewith in order to save space. The inner solution obtained in [1] is of a similar nature except that there a 'ring' vortex is born when motion commences.

The inversion of $\Psi_1{}^{(l,o)}$ requires special attention. Rather than explain it in the body of the text, this is done in Appendix B. The resulting outer disturbance $\psi_1{}^{(l,o)}$ for case (*a*) is shown in Figures 2–5 for $T = 0.1$, 1, 10 and 100. These show that as the cylinder starts moving, the pair of vortices mentioned reappear in the outer field close to the ends of the diameter perpendicular to the stream direction. These migrate away from the obstacle and downstream. So as to get the entire flow pattern in dimensional co-ordinates, one must set $\psi' = Va\,[YRe^{-1} + \Delta Re^{-1}\,\Psi_1{}^{(l,o)}]$, $t = T\nu V^{-2}$, $(x', y') = (X, Y)\nu V^{-1}$. Thus in terms of the scale used in these figures the radius of the cylinder is Re. This basic feature was also obtained in the case of a sphere, except that the vortex core is in the form of a ring. For case (*b*) the inversion and plotting was not carried out.

Appendix A

Here we consider the solution of equation (21) for $\Psi^{(l,o)}$, where $P(S)$ is ζ/S. The l.h.s. of that equation is denoted by $(2/S)\,G(X, Y, S)$, where

$$G(X, Y, S) = \frac{\partial}{\partial Y}\,[\exp{(X/2)}\,K_0(\zeta\sqrt{X^2 + Y^2}\,/2)]. \tag{A.1}$$

Since it follows from equation (19) that $G(X, Y, S)$ satisfies the homogeneous equation (25), a particular solution of the nonhomogeneous equation (21) may be expressed in terms of a series of repeated integrals as follows:

$$\overline{\Psi}_1{}^{(l,o)}(X, Y, S) = -\frac{2}{S}\int_0^X G(X_1, Y, S)\,\mathrm{d}X_1 +$$

$$2\sum_{k=0}^{\infty}(-S)^k\int_0^X\int_0^{X_1}\int_0^{X_2}\cdots\int_0^{X_{k+1}} G(X_{k+2}, Y, S)\,\mathrm{d}X_{k+2}\,\mathrm{d}X_{k+1}\ldots\mathrm{d}X_1. \tag{A.2}$$

Indeed, by operating on this relationship with $\overline{\nabla}^2$ one retrieves equation (21). Defining the X-wise Laplace transform of $\Psi_1{}^{(l,o)}(X, Y, S)$ thus

$$\overline{\Psi}_1^{*(l,o)}(\lambda, Y, S) = \int_0^\infty e^{-\lambda X}\,\overline{\Psi}_1{}^{(l,o)}(X, Y, S)\,\mathrm{d}X, \tag{A.3}$$

and assuming that S is greater than λ, we get from (A.2):

$$\overline{\Psi}_1^{*(l,o)}(\lambda, Y, S) = -\frac{2}{S}\sum_{k=0}^{\infty}\left(-\frac{S}{\lambda}\right)^k G^*(\lambda, Y, S) = -\frac{2}{S}\,\frac{1}{S+\lambda}\,G^*(\lambda, Y, S). \tag{A.4}$$

The inversion of (A.4) is achieved by employing the convolution theorem and the result is

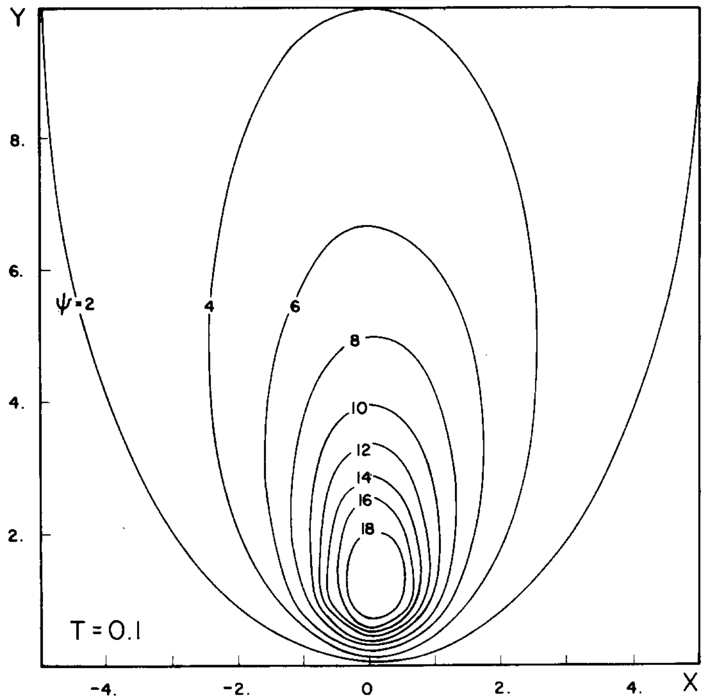

Figure 2. The timewise development of the disturbance flow away from the cylinder at $T = 0.1$.

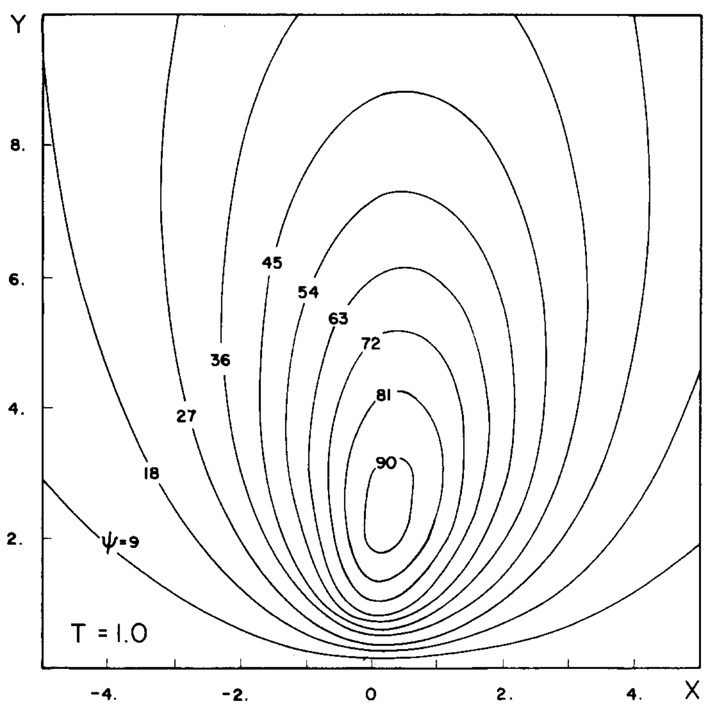

Figure 3. The timewise development of the disturbance flow away from the cylinder at $T = 1.0$.

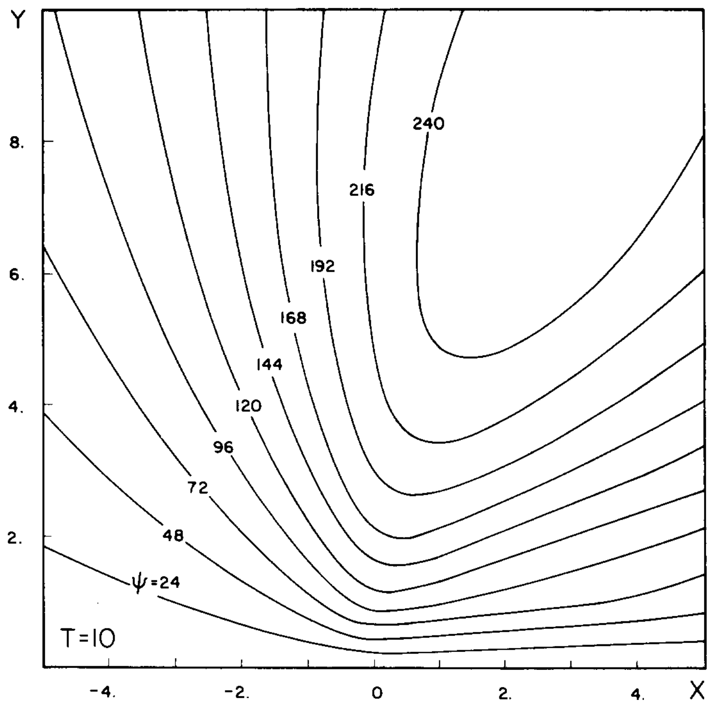

Figure 4. The timewise development of the disturbance flow away from the cylinder at $T = 10$.

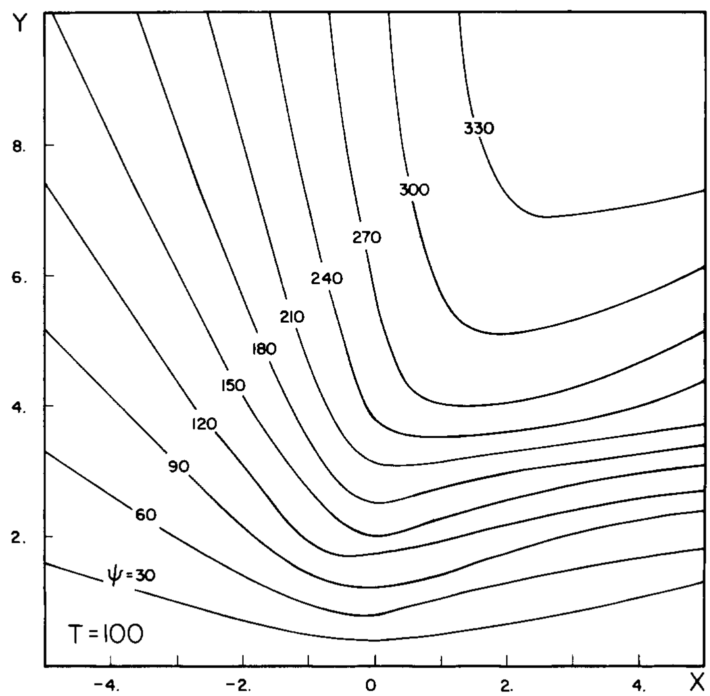

Figure 5. The timewise development of the disturbance flow away from the cylinder at $T = 100$.

$$\overline{\Psi}_1{}^{(l,o)}(X,Y,X) = -\frac{2}{S}\int_0^{\infty} e^{-S\xi}\, G(X-\xi, Y, S)\, d\xi. \tag{A.5}$$

This particular solution of equation (21) is singular at the origin and is discontinuous across the X-axis. Hence, in order to make the solution of that equation well behaved throughout, we express it as follows:

$$\Psi_1{}^{(l,o)}(X,Y,S) = \overline{\Psi}_1{}^{(l,o)}(X,Y,S) + \widetilde{\Psi}_1{}^{(l,o)}(X,Y,S), \tag{A.6}$$

where $\widetilde{\Psi}_1{}^{(l,o)}$ is a solution of the homogeneous equation (21). Clearly $\Psi_1{}^{(l,o)}$ satisfies

$$\Psi_1{}^{(l,o)}(X,Y,S) \to 0, \qquad \text{as} \qquad X^2+Y^2 \to \infty; \tag{A.7}$$

$$\Psi_1{}^{(l,o)}(X,O,S) = 0. \tag{A.8}$$

Taking the limit of (A.5) as $Y \to 0$ one gets

$$\overline{\Psi}_1{}^{(l,o)}(X,O,S) = \frac{2}{S}\lim_{\substack{Y\to 0\\ \epsilon\to 0}}\int_{-\epsilon}^{\epsilon} e^{S(\lambda-X)+\lambda/2}\ln\sqrt{\lambda^2+Y^2}\, d\lambda = \frac{2\pi}{S}\, e^{-SX}. \tag{A.9}$$

Since $\widetilde{\Psi}_1{}^{(l,o)}(X, Y, S)$ is a harmonic function we choose to represent it by a distribution of doublets of strength $\mu(X, s)$ on the positive X-axis. Hence it is given by

$$\widetilde{\Psi}_1{}^{(l,o)}(X,Y,S) = -\frac{\partial}{\partial Y}\int_0^{\infty} \mu(\zeta,S)\ln\sqrt{(X-\zeta)^2+Y^2}\, d\zeta, \tag{A.10}$$

and the Gauss flux theorem implies that the following holds

$$\widetilde{\Psi}_1{}^{(l,o)}(X,O,S) = -\pi\mu(X,S). \tag{A.11}$$

In view of (A.9) we select the following doublet distribution $\mu(X, S) = (2/S)e^{-SX}$ and thus equation (24) is obtained.

Appendix B

In order to find the inverse of the solution of (24) we make use of the following well-known relationships:

$$L^{-1}\left(\frac{e^{-SX}}{S}\right) = H(T-X), \tag{B.1}$$

$$L^{-1}(K_0(\zeta\widetilde{R}/2)) = \frac{1}{2T}\exp\left[-\frac{1}{4}(T+\widetilde{R}^2/T)\right]. \tag{B.2}$$

When used together these render the convolution form

$$L^{-1}\left(\frac{e^{-S\xi}}{S}K_0(\zeta\widetilde{R}/2)\right)=\frac{1}{2}\int_0^{T-\xi}\exp\,[-\frac{1}{4}\widetilde{R}(\tau/\widetilde{R}+\widetilde{R}/\tau)]\,\tau^{-1}\,\mathrm{d}\tau$$

$$=\frac{1}{2}\int_{-\infty}^{\bar{\phi}}\exp\,(-\frac{1}{2}\widetilde{R}\cosh\phi)\,\mathrm{d}\phi, \tag{B.3}$$

where

$$\bar{\phi}=\log\,[(T-\xi)/\widetilde{R}],\qquad \phi=\log\,(\tau/\widetilde{R}). \tag{B.4}$$

The inverse of (24) may be written as

$\psi_1{}^{(l,o)}\,(X,Y,T)=$

$$\frac{\partial}{\partial Y}\int_0^{\infty}\int_{-\infty}^{\bar{\phi}}\exp\left[\frac{1}{2}\widetilde{X}-\frac{1}{2}\widetilde{R}\cosh\phi\right]\mathrm{d}\phi\,\mathrm{d}\xi+2\tan^{-1}\left(\frac{X-T}{Y}\right)-2\tan^{-1}\left(\frac{X}{Y}\right). \tag{B.5}$$

It can also be shown that (B.5) vanishes on the X-axis. For large time this equation reduces to

$$\psi_1{}^{(l,o)}\,(X,Y,\infty)=-2\,\frac{\partial}{\partial Y}\int_0^{\infty}\exp\,(\widetilde{X}/2)\,K_0(\widetilde{R}/2)\,\mathrm{d}\xi-2\tan^{-1}\left(\frac{X}{Y}\right)-\pi \tag{B.6}$$

which, because of relationship (A.9), again vanishes on the X-axis. The velocity field of the steady Oseen flow past a cylinder, as given in Van Dyke's [11] text, may be obtained directly from (B.6) by differentiation.

Appendix C

Here we are concerned with the solution of the integral equation

$$\int_{-\infty}^{\infty}F(\beta)\,Ai\left(\frac{\beta-2S}{2^{1/3}}\right)\mathrm{d}\beta=3/S^2 \tag{C.1}$$

for the unknown function $F(\beta)$. Letting $q=2^{1/3}S$ and $P=2^{-1/3}\beta$ equation (C.1) may be re-written as

$$\int_{-\infty}^{\infty}F(P)\,Ai\,(P-q)\,\mathrm{d}P=6/q^2, \tag{C.2}$$

so that its left-hand side is in the form of the convolution integral. The convolution theorem then readily yields

$$L^{-1}(F(\lambda))\,L^{-1}(Ai(-\lambda)) = -3\,|\lambda|, \tag{C.3}$$

where L^{-1} denotes the inverse complex Fourier transform. To find the inverse transform of the Airy integral, we employ the integral representation of the Airy function (Jeffreys and Jeffreys [5]) yielding

$$\begin{aligned} L^{-1}(Ai(-\lambda)) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} Ai(-\eta) \exp(i\lambda\eta)\, d\eta = \\ &= (2\pi)^{-3/2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \cos\left(\frac{1}{3}\sigma^3 - \eta\sigma\right) \exp(i\lambda\eta)\, d\sigma\, d\eta, \end{aligned} \tag{C.4}$$

which can be also expressed as

$$L^{-1}(Ai(-\lambda)) = (2\pi)^{-3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(i\frac{1}{3}\sigma^3 - i\eta\sigma + i\eta\lambda\right) d\sigma\, d\eta. \tag{C.5}$$

Recalling the integral representation of the Dirac delta function

$$(2\pi)^{-1} \int_{-\infty}^{\infty} \exp(i\gamma\eta)\, d\eta = \delta(\gamma), \tag{C.6}$$

equation (C.5) renders a closed form solution:

$$L^{-1}(Ai(-\lambda)) = (2\pi)^{-1/2} \exp\left(i\frac{1}{3}\lambda^3\right) \tag{C.7}$$

Thus, the function $F(P)$ by equation (C.3) is given by

$$F(P) = -3 \int_{-\infty}^{\infty} |\lambda| \exp\left(i\frac{1}{3}\lambda^3 + i\lambda P\right) d\lambda \tag{C.8}$$

or

$$F(\beta) = -3 \cdot 2^{2/3} \int_{-\infty}^{\infty} |\xi| \exp\left(i\frac{2}{3}\xi^3 + i\xi\beta\right) d\xi\,. \tag{C.9}$$

To verify that indeed (C.9) is a solution of (C.1) we substitute the latter in the former yielding

$$\begin{aligned} &-\frac{6}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\xi| \exp\left(i\frac{2}{3}\xi^3 + i\xi\beta + i\frac{2}{3}\eta^3 + i\eta(\beta - 2S)\right) d\xi\, d\eta\, d\beta \\ &= -6 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\xi| \exp\left(i\frac{2}{3}\xi^3 + i\frac{2}{3}\eta^3 - 2i\eta S\right) \delta(\xi + \eta)\, d\xi\, d\eta \\ &= -6 \int_{-\infty}^{\infty} |\eta| \exp(-2i\eta S)\, d\eta = 3/S^2 \end{aligned} \tag{C.10}$$

as we wished to show. Again, the last integral in (C.10) is also interpreted in Lighthill's [6] generalized-function sense.

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